Hans-Jakob Schindler [1]

EXPERIMENTAL DETERMINATION OF CRACK CLOSURE BY THE CUT COMPLIANCE TECHNIQUE

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ABSTRACT: The cut compliance method was developed to measure residual stress profiles. Its idea is to release them by introducing progressively a cut into the considered body. From the strain change due to progressive cutting it is possible to calculate the distribution of the released stresses. Since the method uses fracture mechanics principles the stress intensity factor due to residual stresses is delivered as well. Being a special type of residual stresses, the distribution of the contact stresses of fatigue crack closure and the corresponding closure stress intensity factors can be suitably determined by this method. Furthermore, it enables the residual stresses in front of the crack-tip to be determined, which in crack retardation play a role as important as the closure effects. The basic theoretical relations for some frequently used fatigue specimens are analytically derived. They also can serve to evaluate the contact stresses from the commonly used unloading compliance technique, that is often used to identify crack closure effects. Some preliminary examples demonstrate the capability of the suggested method.

KEYWORDS: Crack closure, closure stress intensity factor, experimental, influence functions, residual stress

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The phenomena associated with crack closure result from contact stresses between the crack faces of a fatigue crack in the vicinity of its tip. Being in self-equilibrium, these stresses can be considered as a special type of residual stresses, so in principle methods to determine the latter can also be used to determine closure stresses. A suitable mechanical method to measure the distribution of residual stresses has been suggested and developed by Cheng and Finnie, the so-called crack compliance method [1, 2] (see [2] for further references). A similar procedure was independently used also by Fett [3] and by Kang et al. [4]. Its basic idea is to introduce progressively, step by step, a cut along the plane where the residual stresses are to be measured, and to record the change in strain at a suitable

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[1] Senior research engineer, Swiss Federal Labs. for Materials Testing and Research (EMPA), CH-8600 Duebendorf, Switzerland.

location, where the strain is affected by progressive cutting. Therefrom, the released stresses can be calculated. To establish the corresponding mathematical equations, some basic relations of linear elastic fracture mechanics can be utilized, which simplifies the analysis. Recently the cut compliance method (CC-method) was extended by the present author and co-workers such that it became - as will be shown herein - a particularly well-suited tool to determine fatigue crack closure effects: Firstly, the possibility to determine directly the stress intensity factors (SIF) due to the residual stresses was introduced [5, 6], secondly, an incremental forward inversion technique was introduced that is able to deal even with steep stress gradients without convergence problems [7, 8], and thirdly, closed-form solutions were derived for the required relation between the strain change at the rear edge of the plate and the stress intensity factor (SIF) due to the residual stress at the cut tip [9], particularly for cases of relatively deeply notched plates [10]. (N.B. To avoid confusions with other compliance methods that are often used to measure closure effects [11, 12, 13], the term "cut compliance method" is used in the present paper instead of "crack compliance method" as in most of the given references)

In principle the idea of introducing a cut into the closure zone of a fatigue crack to release and measure the closure stresses is not new; it already was used in [14] to get information about closure effects. The objective of the present paper rather is to show how to adapt and to use the CC-method for this purpose, and to point out its experimental and theoretical advantages regarding simplicity, sensitivity and accuracy. Being based on exact analytical solutions and requiring only one strain gage measurement as input-data, it is experimentally rather simple as well as theoretically accurate. In the following the basic theory of the method will be recapitulated and the relevant influence functions for cracked specimens as used in fatigue testing are derived. Furthermore the relation of the CC-method to other unloading compliance techniques as often used in fatigue is worked out. The suitability and capability of the method is demonstrated by some experimental results.

THE PRINCIPLE OF THE CC-METHOD

A cut has to be progressively introduced along the plane where the residual stress are to be measured, which in the case of closure stresses means in the plane of the fatigue crack. Fig. 1 shows exemplarily the case of a rectangular plate containing a fatigue crack of a length a_f, which was initiated at the tip of a notch of length a_n. The actual length of the cut is denoted by “a“. In an overall view, the cut can be considered as a perfect crack (see discussion at the end of this paper), so the basic equations of linear elastic fracture mechanics still hold and can serve to establish the required mathematical relation between the residual stress (in the present paper this term also covers contact stresses in the closure area) and the strain ε_M at the measurement point M. It can be easily shown [5] that the SIF at the tip of the cut

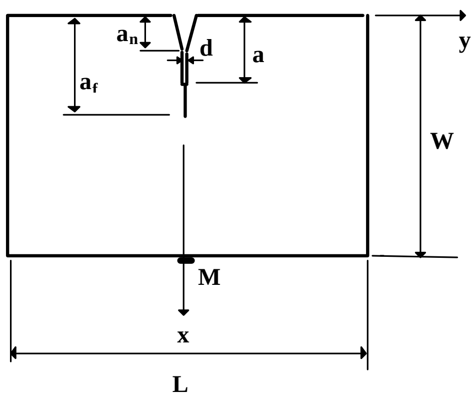


Fig. 1: A cut of length a introduced in the plane of a fatigue crack of length a_f

due to the residual stresses (we restrict ourselves to Mode-I, which is predominating in the case of closure stresses) is given by

$$K_{Irs}(a) = \frac{E'}{Z(a)} \frac{d\varepsilon_M}{da} \tag{1}$$

where E' denotes the generalized Young's modulus (i.e. E'=E for plane stress and E'=E/(1-ν^2) for plane strain) and Z(a) the so-called influence function. The latter is a unique function that depends on the component geometry, on the cut plane and on the location of the strain gage, but not on the residual stress distribution. To evaluate (1), $\varepsilon_M(a)$ has to be recorded during the cutting process as a function of the cut depth. In principle the measurement point M is arbitrary, but there are large differences regarding its sensitivity, which is characterized by the absolute value of Z(a). Usually the most sensitive location is the one on the rear surface's intersection with the crack plane, as indicated in Fig. 1. Determination of Z(a) is the crucial and - with regard to the theoretical and computational effort - the most demanding step of the CC-method [4, 5, 8]. However, Z(a) needs to be determined only once for a certain geometry and measurement location. In the next section, solutions of Z(a) for some specimen shapes that are commonly used in fatigue testing are presented.

The cut width d should be chosen as small as possible. However, as long as a>>d and (W-a)>>d the effect of the correspondingly blunt notch tip (compared with the theoretically required crack) is negligible. One can show that the energy released due to widening of a narrow crack to a notch of width d is of the same order of magnitude as due to a crack prolongation of the amount $\Delta a=d$. Thus, the finite cut width is just averaging the resulting stresses over a distance of about $\Delta x=d$, which is acceptable since in 2D they are averaged over the specimen thickness B anyway.

If $a<a_f$, $K_{Irs}(a)$ as delivered by (1) represents the SIF due to the closure stresses. Its value at $a= a_f$ represents the well known quantity closure SIF K_{cl},

$$K_{cl} = K_{Irs}(a=a_f) \tag{2}$$

For $a>a_f$, the effect of the residual stresses in front of the crack-tip is included in $K_{Irs}(a)$ as well.

From the experimentally determined function $K_{Irs}(a)$ it is possible to calculate the initial distribution $\sigma_{rs}(x)$ of the residual normal stresses (which for $a<a_f$ is the distribution of the contact pressure) by inversion of the general relation

$$K_{Irs}(a) = \int_0^a h(x,a) \cdot \sigma_{rs}(x) \cdot dx \tag{3}$$

where h(x,a) denotes the so-called weight function as introduced by Bueckner [15]. Formulas of h(x,a) for the specimen geometries considered here can be found in [16]. The inversion of (3) can be achieved by the step-by-step-procedure suggested in [8] and summarized below.

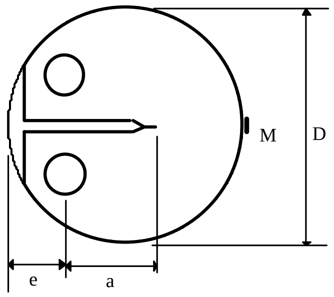
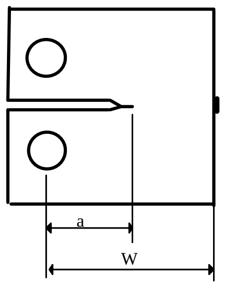
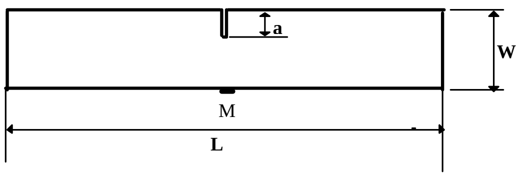
a) Disk-shaped CT-specimen  	$$Z(a) = -\frac{7.952}{p \cdot D^2} \cdot \sqrt{\frac{a+e}{\left(1-\frac{a+e}{D}\right)^3}} \qquad (5)$$
b) Standard CT-specimen  	$$Z(a) = \frac{-2.532}{(W-a)^{3/2}} \qquad (6)$$ (for a>0.5W)
c) Beam-shaped specimen (L>2W)  	$$Z(a) = \frac{-2.532}{(W-a)^{3/2}} \sqrt{1-25 \cdot \left(a/W - 0.2\right)^2} \cdot \left[5.926 \cdot \left(0.2 - a/W\right)^2 - 0.288 \cdot (0.2 - a/W) + 1\right]$$ (for a/W≤0.2) (7a) $$Z(a) = \frac{-2.532}{(W-a)^{3/2}}$$ (for 0.2<a/W<1) (7b)

Fig. 2: Three specimen types often used in fatigue testing, and corresponding influence functions Z(a)

INFLUENCE FUNCTIONS

Being a unique function independent on the stress distribution, Z(a) as required in (1) can be calculated by considering an arbitrary load case, the so-called reference load case, by

$$Z(a) = \frac{E'}{K_{Iref}(a)} \cdot \frac{de_{Mref}}{da}(a) \qquad (4)$$

where $K_{Iref}(a)$ and $\varepsilon_{Mref}(a)$ denote the stress intensity factor and the strain at M, respectively, for the reference load case [5, 6]. In general, numerical methods, e.g. the finite element method (FEM), have to be used to evaluate (4). In the following, closed form solutions for Z(a) are given for three crack geometries that are often used in fatigue testing (Fig. 2): The formulas (5), (6), and (7b), which are "exact" solutions, are derived in the appendix. Eq. (7a) is an approximation obtained from FEM - results and curve-fitting .

DISTRIBUTION OF CLOSURE STRESSES

From $K_{Irs}(a)$ as obtained experimentally by using (1), it is possible to calculate the residual stress distribution $\sigma_{rs}(x)$ by inversion of eq. (3), which can be performed according to [8] as follows. The residual stress distribution $\sigma_{rs}(x)$ is approximated by a series of small steps as shown schematically in Fig. 3, so the stress level at each step can be calculated by applying eq. (3) to a hypothetical, incrementally prolonging crack. The stress level of the first increment, σ_0, which represents the average stress acting near the front surface in the range $0<x<a_0$ (where $a_0 << W$ should be fulfilled) is obtained from the well known relation between the stress and the SIF of a short edge crack, i.e.:

$$\sigma_0 = \frac{K_{Irs}(a_0)}{1.12 \cdot \sqrt{\pi \cdot a_0}} \tag{8}$$

If the fatigue crack is - as usual - initiated by a starter notch of depth a_n, then a_0 and σ_0 shall be chosen $a_0=a_n$, and $\sigma_0= 0$, respectively, since there are no closure stresses on the initial notch. In order to calculate the average stress level σ_1 of the next step (i.e. the average stress in the range $a_0<x<a_0+\Delta a$), we extend the hypothetical crack by the increment Δa and apply eq. (3). Then a further crack increment is assumed, for which the corresponding stress level follows again from eq. (3), and so forth across the whole cross section. Denoting the length of the hypothetical crack after i prolongation increments Δa by a_i (i.e., $a_i=a_0+i\cdot\Delta a$), and the average stress in the corresponding interval $a_{i-1}<x<a_i$ by σ_i (see Fig. 3), eq. (3) is approximated by

$$K_{Irs}(a_i) = \sigma_0 \cdot \int_0^{a_0} h(x,a_i)\cdot dx + \sum_{j=1}^{i-1} \sigma_j \cdot \int_{a_{j-1}}^{a_j} h(x,a_i)\cdot dx + \sigma_i \cdot \int_{a_{i-1}}^{a_i} h(x,a_i)\cdot dx \tag{9}$$

which allows σ_i to be calculated for each step. The resulting step distribution converges to the exact solution $\sigma_{rs}(x)$ as $\Delta a \rightarrow 0$. Some difficulties (mathematical instabilities) may arise in the region near the rear surface (i.e. for W-a<<W), because weight functions, which usually are approximations, might be not accurate enough in this range [7]. For this reason it is recommended to replace for cut depths of about $a_i>0.8W$ eq. (9) by

$$K_{Irs}(a_i) = \sum_{j=0}^{i} \left\{ \frac{3.97\sigma_j \cdot \Delta a \cdot \left[0.264W + 0.736a_i - (a_0 + j\cdot\Delta a)\right]}{(W-a_i)^{3/2}} + \frac{1.46\sigma_j \Delta a}{(W-a_i)^{1/2}} \right\} + 2\sigma_i \sqrt{\frac{2}{\pi} q \cdot (W-a_i)} \tag{10}$$

as explained in [8]. The non-dimensional factor q, which is about 0.03, can be determined from the condition that transition from eq. (9) to eq. (10) at the chosen transition cut depth $a=a_i$ shall deliver a continuous function $\sigma_{rs}(x)$ at $x=a_i$. Using eq. (10) guarantees that the condition of self-equilibrium of the calculated residual stresses will be fulfilled.

CLOSURE STRESSES FROM THE UNLOADING COMPLIANCE

It is well known that the slope of the force-vs.-displacement or -vs.-strain curve of a specimen that contains a crack is changing when an increasing or decreasing external force is applied. This effect is due to the geometrical nonlinearity resulting from the load-dependence of the contact area in the closure zone, which acts like a changing crack length. If the load F is increasing, the effective crack length seems to grow from $a=a_{cl}$ at F=0 to $a=a_f$ at $F=F_{op}$, where F_{op} is the so-called opening load and a_f the actual length of the fatigue crack. Regarding the compliance in the sense of the CC-method, a continuous loading has the same effect as a continuous cutting. Thus, instead of cutting an increasing external load can be applied, provided the required strain-vs.-force curve is accurate enough to exhibit the changes in the slope of the curve, which are small. To avoid interference with plastic effects which also contribute to nonlinearities, unloading instead of loading is preferable. As shown below, the analysis of the CC- method can be easily extended to make it applicable to cracks that are extended or contracted by external forces.

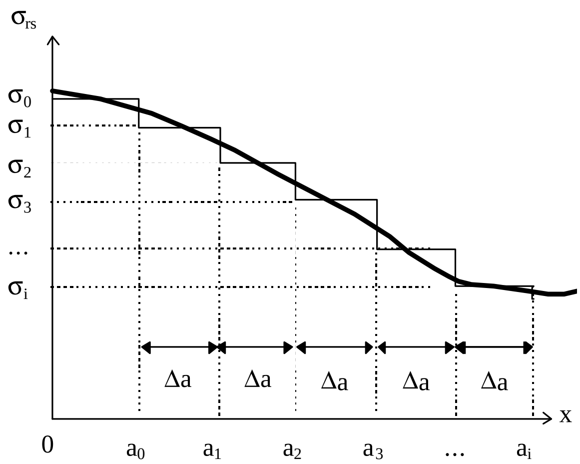


Fig. 3: Step-wise determination of the residual stress distribution $\sigma_{rs}(x)$

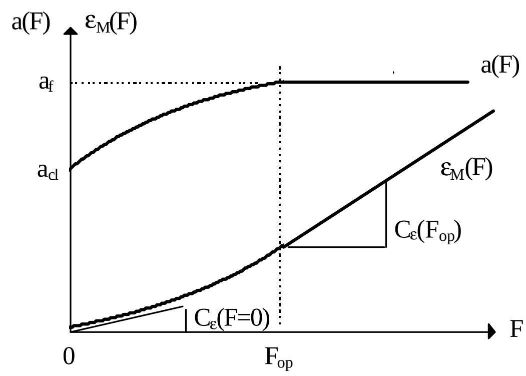


Fig. 4: Behaviour of the effective crack length a and the strain for increasing or decreasing external load F (schematic)

The strain that results at M due to the load F and the increasing effective crack length, a, which in the considered case means the length of the contact-free zone of the crack faces, follows readily from (1) to be

$$\varepsilon_M(a\ ,F)=\varepsilon_M(a_{cl})+\frac{1}{E'}\cdot\int_{a_{cl}}^{a}K_I(F,a)\cdot Z(a)\cdot da \qquad (11)$$

The compliance with respect to the strain at M is obtained therefrom to be

$$C_e(a) = \frac{d\varepsilon_M(a, F)}{dF} = C_e(F=0) + \frac{1}{E'} \cdot \int_{a_{cl}}^{a} k_I(a) \cdot Z(a) \cdot da \qquad (12)$$

where $k_I(a) = K_I(a)/F$. By (12), the actual effective crack length, a, corresponding to the actual load, F, can be obtained from the actual slope of the curve $\varepsilon_M(F)$ (Fig. 4) . Once knowing the effective crack length, the corresponding SIF, which is equal to the SIF due to the contact stresses, thus equivalent to $K_{Irs}(a)$ as given in (1) for $a_{cl}<a<a_f$, follows from

$$K_I(a, F) = F \cdot k_I(a) = K_{Irs}(a) \qquad (13)$$

From $K_{Irs}(a)$ as given by (13) the distribution of the closure stresses can be obtained as described in the previous chapter. The total closure SIF is $K_{cl} = K_{Irs}(a_f)$.

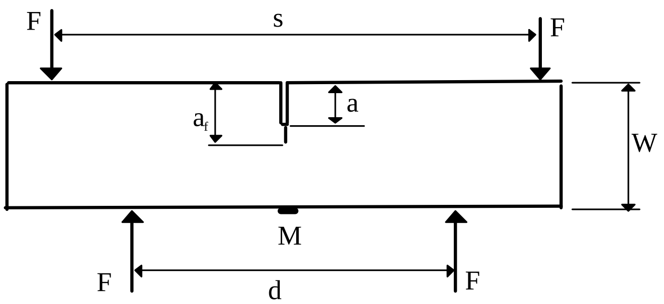


Fig. 5: Edge crack in a beam opened by 4-point bending

As an example we consider the case of a relatively deeply notched beam or rectangular plate (Fig. 5, a>0.25W), subjected to 4-point-bending as external forces to open the crack. The SIF for this case can be approximated by [20]

$$K_{Irs}(a) = \frac{3.97 \cdot F \cdot (s-d)}{2B \cdot (W-a)^{1.5}} \quad \text{thus:} \quad k_I = \frac{1.98 \cdot (s-d)}{B \cdot (W-a)^{1.5}} \qquad (14)$$

Z(a) for this case is given by (7b). Therewith, (12) leads to

$$\Delta C_e(F, F_{op}) = \frac{5.026}{E' \cdot B \cdot W^2} \cdot \left[\left(1 - \frac{a_f}{W}\right)^{-2} - \left(1 - \frac{a}{W}\right)^{-2} \right] \qquad (15)$$

where $\Delta C_\varepsilon(F, F_{op}) = C_\varepsilon(F_{op}) - C_\varepsilon(F)$ denotes the difference of the slope of the force-vs.-strain curve between the opening load F_{op} and the actual load F. This equation can easily be solved for the actual crack length a,

$$\frac{a}{W}(F) = 1 - \sqrt{\left[\left(1 - \frac{a_f}{W}\right)^{-2} - \frac{\Delta C_e(F, F_{op}) \cdot E' \cdot B \cdot W^2}{5.026} \right]^{-1}} \qquad (16)$$

By this equation, the actual length of the contact area can be determined from the experimentally determined change in the slope of the compliance curve and the optically measured a_f . Together with (14) it is possible to get $K_{Irs}(a)$, i.e. the same information as given by (1), but limited to the closure zone $a_{cl}<a<a_f$.

EXAMPLES

Two CT-specimens of W=50mm made of a mild austenitic stainless steel (R_p=200MPa) were fatigued in accordance with the standard ASTM E813. The initial notch depth was $a=a_n=27$ mm, the fatigue crack length $a=a_f\approx30$mm. The cut was produced by electrical discharge machining (EDM). The SIF resulting from (1) and (6) for specimen S1 is shown in Fig. 6. The stresses calculated from the SIF-distribution by (8) and (9) are shown in Fig. 7. The second specimen, E1, was additionally subjected to an overload up to the plastic range. The measured SIF and stresses are also shown in Fig. 6 and 7. Comparing the results of these two experiments shows that the contact pressure that is present in specimen S1 in the range 27.5mm < a< 30mm, has vanished in the specimen E1 due to the overload. On the other hand, the overload significantly increased the residual stresses in the ligament. This corresponds qualitatively to the expected behaviour. There are two further theoretically expected features of these curves: First, the SIF due to residual stresses (Fig. 6) should tend to zero as the cut depth a approaches the rear surface and second, the residual stresses (Fig. 7) should be in self-eqilibrium. Just by visual inspection of the corresponding curves these requirements seem to be fulfilled, which is another indication that the method works properly and the results are reliable.

To further explore the capability of the CC-method it was applied to a short surface crack. For this purpose, a beam-shaped specimen of W=13.5mm with a starter notch of a depth a_n = 1.5 mm was loaded in cyclic bending until a crack of about 0.5mm length was formed at the notch. Then a surface layer as thick as the depth of the notch was removed, so a flat beam-shaped specimen of W =12mm containing an edge crack of about 0.5 mm depth remained. The resulting SIF-distribution obtained from (1) and (7a) is shown in Fig. 8. Fig. 9 shows the corresponding contact pressure and residual stress distribution. Obviously such special problems can also be handled by the CC-method.

DISCUSSION AND CONCLUSIONS

It is shown that the cut compliance technique offers the possibility to determine experimentally crack closure effects, which includes the distribution of contact stresses in the closure zone of the crack surface, the corresponding stress intensity factor as well as the residual stresses in front of the crack tip. Experimentally, this method is remarkably simple. One just needs to introduce progressively a narrow cut along the crack plane and to measure the resulting strain change at the back surface of the specimen. The slope of the strain-vs.-cut depth curve is proportional to the SIF due to the closure stresses, with the so-called influence function representing the corresponding linear relation. Though exact, the latter is particularly simple for the cases of deeply notched specimens that are common in fatigue testing. Thus this method represents a very suitable means to identify crack closure effects and to investigate related crack retardation mechanisms.

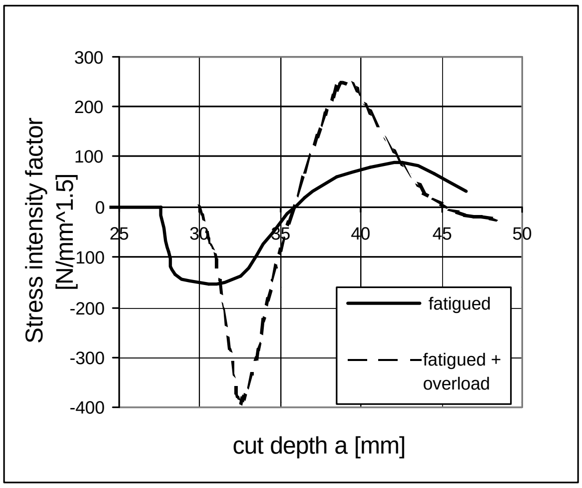


Fig 6: SIF as a function of cut depth a for two CT-specimens: Specimen S1 (fatigued) and specimen E1 (with an additional single overload)

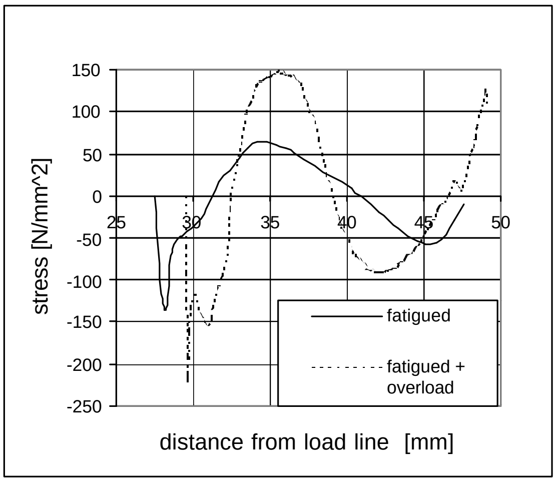


Fig. 7: closure and residual stresses as a function of the distance from the load line (see Fig. 2) corresponding to SIF as given in Fig. 6

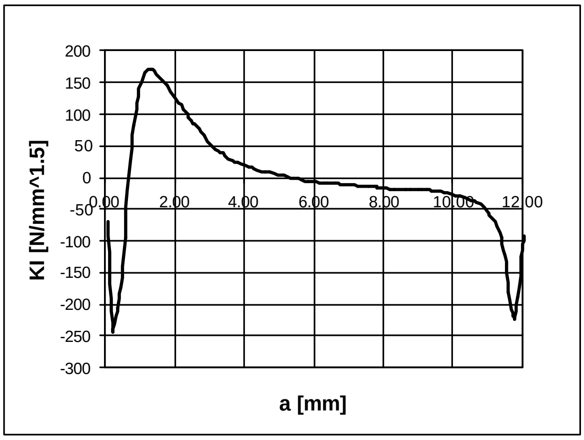


Fig. 8: Stress intensity factor due to closure and residual stresses in the case of a beam containing a short surface crack

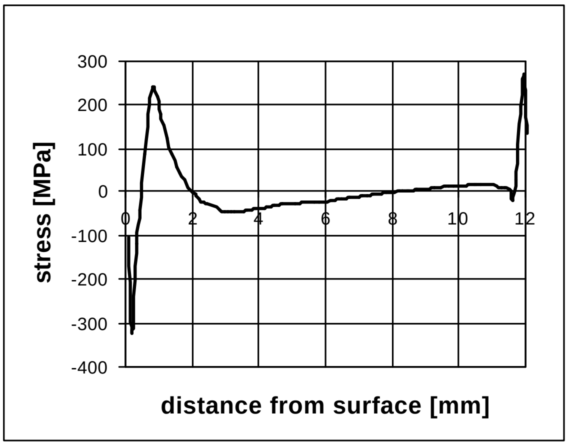


Fig. 9: Closure stress and residual stress corresponding to Fig. 8

Crack retardation effects are often categorized as so-called extrinsic and intrinsic. Crack closure belongs to the former, whereas residual stresses in front of the crack tip to the latter. However, as shown by the exemplary experimental results (Fig. 6 - 9), the contact stresses in the wake of the crack and the residual stresses in front of the crack tip follow a continuous curve, as does the resulting SIF. Both curves exhibit no indication of the exact position of the crack tip. The closure SIF is defined as the value of the SIF-curve at the tip of the fatigue crack. In general, the latter is hard to define, since the crack front in general does not form a straight line. Thus, it is hardly possible to unambiguously distinguish between crack closure in the wake, on one hand, and residual stresses in front of the crack tip on the other. This implies that a strict distinction between these two categories, which from a theoretical point of view is helpful, from a physical point of view is less relevant. It seems that they should rather be considered as interacting mechanisms, which have a similar effect on the further crack propagation. For instance compare the results of the CT-specimens with and without overload (Fig. 6 and 7): The overload of specimen E1 obviously wiped out the fatigue closure stresses, but increased the residual stresses in front of the crack. It is well known that such an overload causes additional crack retardation. Actually, the residual stress ahead of the crack tip reflects, to a certain extent, the future closure stresses that the crack will feel as it grows through the zone of compressive residual stresses.

In this sense, being able to deliver the corresponding experimental data, the CC-method applied to fatigue cracks will probably open some new insight in crack retardation mechanisms.

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APPENDIX

A. Determination of Influence Functions

Generally, the influence function as defined in (1) can be obtained by (3), i.e.

$$Z(a) = \frac{E'}{K_{Iref}(a)} \cdot \frac{d\varepsilon_{Mref}}{da}(a) \tag{A1}$$

where $K_{Iref}(a)$ and $\varepsilon_{Mref}(a)$ denote the stress intensity factor and the strain at M, respectively, for the considered reference load case. In the following the solutions for a circular disk containing a radial edge crack (Fig. A.1) and a deeply edge-cracked plate or beam (Fig. A2) are derived, which form the basis of the formulas presented in this paper.

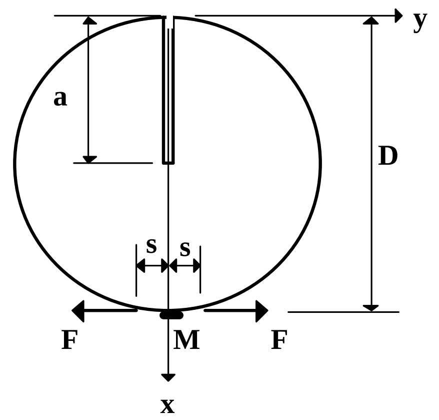

Fig. A.1: Circular disk containing an edge crack

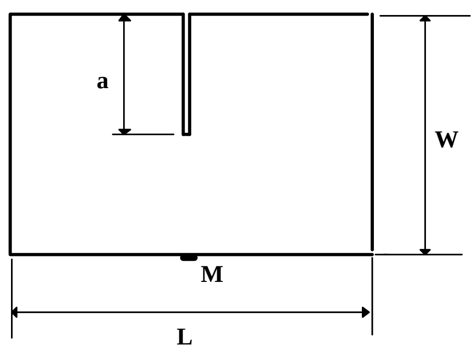

Fig. A.2: Deeply cracked rectangular plate

A.1 Circular Disk

Consider a radially cracked circular disk (Fig. A.1). A suitable reference load case is a uniformly pressure p acting on the crack surface. For this load case the exact solution for the SIF is given in [17, 18] to be

$$K_{Iref} = 1.988 \cdot p \cdot \sqrt{\frac{a}{(1 - a/D)^3}} \tag{A2}$$

According to Castigliano's principle [19], the reference strain $\varepsilon_{Mref}(a)$ is obtained by

$$\varepsilon_M = \frac{1}{2} \frac{\partial^2 U}{\partial F \partial s} \bigg|_{\substack{F=0 \\ s=0}} \tag{A3}$$

where F denotes a pair of virtual forces acting at y = ±s from M, as shown in Fig. A.1, and U the strain energy, which is

$$U = \frac{B}{E'} \int_0^a \left[K_{Irs} + K_{IF} \right]^2 da \tag{A4}$$

K_{IF}, stress intensity factor due to the virtual forces F, can be written as

$$K_{IF}(s) = \int_0^a h(x,a) \cdot \sigma_{yF}(x,s) \cdot dx \tag{A5}$$

where $\sigma_{yF}(x,s)$ is the distribution of the normal stresses along the x-axis due to F in the absence of a crack. It is obtained from the general exact solution given in [5] to be

$$\sigma_{yF}(x,s) = \frac{2F}{\pi \cdot B} \cdot \left\{ \frac{s^3}{\left[(D-x)^2 + s^2\right]^2} - \frac{2s}{D^2} \right\} \tag{A6}$$

Using (A1) and (A3) - (A6) one finds

$$Z(a) = -\frac{4}{\pi \cdot D^2} \int_0^a h(x,a) dx \tag{A7}$$

The same integral appears when the SIF for the reference load case of a homogeneous pressure acting on the crack surface is calculated by the weight function technique (see (A4)). From comparison with (A2) one finds

$$\int_0^a h(x,a) dx = 1.988 \cdot \sqrt{\frac{a}{(1 - a/D)^3}} \tag{A8}$$

Inserting (A8) in (A7) leads to

$$Z(a) = -\frac{7.952}{\pi \cdot D^{3/2}} \cdot \sqrt{\frac{a/D}{(1 - a/D)^3}} \tag{A9}$$

Based on exact solutions, (A9) holds without restrictions for 0<a<D. Note that the crack length, a, is defined somewhat different from Fig. 2a.

A.2. Deeply Cracked Rectangular Plates

Consider an edge-cracked rectangular beam as shown in Fig. 2, loaded as a reference load by a uniform pressure acting on the crack surfaces. In the case of a relatively deep crack, which means for a>0.2W and W-a<0.6L , the dominating geometrical quantity in the SIF as well as the strain at M is the ligament width W-a. Thus, a dimensional analysis leads to

$$Z(a) = -\frac{A}{(W-a)^{3/2}} \tag{A10}$$

where A is a non-dimensional constant, which is determined by the following limit analysis. K_{Iref}, ε_{Mref} as well as Z(a) for the disk (Fig. A.1) and the rectangular plate tend to coincide as a□D and a □W, respectively. Therewith one readily finds

$$Z(a) = -\frac{7.952}{p \cdot (W-a)^{3/2}} \tag{A11}$$